

HIGH-ORDER FINITE ELEMENTS ON PYRAMIDS I: APPROXIMATION SPACES.

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Abstract. We present a family of high-order conforming finite elements on pyramids, which have polynomial traces onto the boundary of the pyramid. We first show that it is not possible to use approximation spaces consisting purely of polynomials. We then introduce an infinite reference pyramid and describe (via pullbacks) analogs of the usual Sobolev spaces on it. Using the hexahedral nature of this element, we are able to construct a family of approximation spaces of arbitrary order. The pullbacks of these onto the finite pyramid provide the requisite approximation spaces. In a companion article [19], we show shape functions and degrees of freedom for these elements, with a view to easy computability. The discrete spaces satisfy an exact sequence property, and contain high-degree polynomials; this is also established in the companion paper.

1. Introduction. High order conforming finite elements for $H(\text{curl})$ and $H(\text{div})$ spaces based on meshes composed of tetrahedra and hexahedra were first presented by Nedelec, [18]. In some circumstances, it may be desirable, or even necessary, to use hybrid meshes which include both tetrahedral and hexahedral elements, see e.g. [4]. If these meshes are to avoid hanging nodes then they will, in general, contain pyramids. A specific example arises from hybrid the application of hybrid Yee / FEM-FDTD schemes for Maxwell's equations, (e.g. [21, 25]) where a FDTD method (using cubes) is used to discretize in sub-regions with uniform coefficients and rectilinear geometric features, and an edge finite element discretization using tetrahedra is used in sub-regions where the fields vary rapidly, or near complex (curvilinear or singular) geometric features. Pyramidal elements are used in a layer between these two meshes.

Let Ω be a pyramid with a square base defined as:

$$\Omega = \{\xi = (\xi, \eta, \zeta) \in \mathbb{R}^3 \mid \xi, \eta, \zeta \geq 0, \xi \leq 1 - \zeta, \eta \leq 1 - \zeta\}. \quad (1.1)$$

It is our aim to construct high order finite elements on a pyramid. Concretely, in this paper we present finite element triples, $(\Omega, \mathcal{U}^{(s),k}(\Omega), \Sigma^{(s),k})$, for positive integers k which are unisolvent conforming finite elements for $H^1(\Omega)$, $H(\text{curl}, \Omega)$, $H(\text{div}, \Omega)$, $L^2(\Omega)$ respectively for $s = 0, 1, 2, 3$. Here $\mathcal{U}^{(s),k}(\Omega)$ denotes the k th order finite dimensional approximation space for the relevant Sobolev space and the sets $\Sigma^{(s),k}$ are the associated degrees of freedom. We seek finite elements with the following properties

P1-P3:

- P1) Compatibility:** Not only should the elements be conforming, but the restriction of each element to its triangular and quadrilateral face(s) should match that of the corresponding canonical tetrahedral and hexahedral finite element. Specifically, what should match are the traces and exterior degrees of freedom. In other words, these elements satisfy the correct *patching conditions* on inter-element boundaries, [12].
- P2) Approximation:** The discrete spaces $\mathcal{U}^{(s),k}(\Omega)$ should allow for high-order approximation to the spaces $H^1(\Omega)$, $H(\text{curl}, \Omega)$, etc. In particular, given a positive integer p , it should be possible to choose k such that all polynomials of degree p are contained in $\mathcal{U}^{(s),k}(\Omega)$.

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P3) Stability: The elements satisfy a commuting diagram property:

$$\begin{array}{ccccccc}
H^r(\Omega) & \xrightarrow{\nabla} & H^{r-1}(\text{curl}, \Omega) & \xrightarrow{\nabla \times} & H^{r-1}(\text{div}, \Omega) & \xrightarrow{\nabla \cdot} & H^{r-1}(\Omega) \\
\Pi^{(0)} \downarrow & & \Pi^{(1)} \downarrow & & \Pi^{(2)} \downarrow & & \Pi^{(3)} \downarrow \\
\mathcal{U}^{(0),k}(\Omega) & \xrightarrow{\nabla} & \mathcal{U}^{(1),k}(\Omega) & \xrightarrow{\nabla \times} & \mathcal{U}^{(2),k}(\Omega) & \xrightarrow{\nabla \cdot} & \mathcal{U}^{(3),k}(\Omega)
\end{array} \quad (1.2)$$

Here $\Pi^{(s)}$, denote interpolation operators induced by the degrees of freedom, $\Sigma^{(s),k}$ and r is chosen so that the interpolation operators are well defined.

Gradinaru and Hiptmair [12] constructed ‘Whitney’ elements satisfying properties **P1** and **P3** and our family of elements includes these as the lowest order case, see the Appendix. In the engineering literature, Zgainski et al [8, 26] appear to have discovered the same first order $H(\text{curl})$ -conforming element independently and also demonstrated a second order element. In [4] the authors describe high-order finite elements for $H^1(\Omega)$, but not the other spaces. Graglia et al [13] constructed $H(\text{curl})$ and $H(\text{div})$ elements of arbitrarily high order. Similarly, Sherwin [22] demonstrated H^1 -conforming elements also satisfying properties (1) and (2). These high order constructions provide an explicit scheme for determining nodal basis functions; none of them address property (3).

The mimetic finite difference method, originally presented in 1997 [14] and further developed by several authors (e.g., [15, 7, 6, 5]) seeks to develop approximations on polyhedral meshes and hence includes pyramids as a special case.

The major results of this paper are: the construction of conforming discrete spaces $\mathcal{U}^{(0),k}(\Omega), \mathcal{U}^{(1),k}(\Omega), \mathcal{U}^{(2),k}(\Omega)$ and $\mathcal{U}^{(3),k}(\Omega)$ for arbitrary (positive integer) order k . We show that these spaces admit convenient Helmholtz-like decompositions, and that their traces on faces and edges are consistent with traces from neighbouring elements. Hence property **P1** is satisfied.

In a companion article, [ref] we provide a description of the degrees of freedom, $\Sigma^{(s),k}$ and demonstrate unisolvency. The exterior degrees of freedom agree precisely with those specified by neighbouring tetrahedral or hexahedral elements. Properties **P2** and **P3** are also established in the companion article, as are explicit shape functions for these high-order finite elements. To foreshadow that discussion, we will use the projection-based interpolation described in [9, 10] to solve the difficult problem of defining the internal degrees of freedom on a pyramid. It is possible to use projection based interpolation for the external degrees too, and we believe that the hp framework of which it is a part will also accommodate our element. However, this is not our immediate objective and the external degrees described in [17] allow for a more explicit exposition.

The use of rational functions to construct finite element approximations is not new. In [23] Wachspress describes their use for regions in the plane bounded by line segments and arcs. Several previous constructions of finite elements on a pyramid use rational functions, [13, 22, 8, 26]. All of these use non-polynomial shape functions directly on the pyramid, possibly inspired by the Duffy transformation mapping the cube to the pyramid. In contrast, we use a non-linear (projective) mapping of the reference pyramid onto an infinite pyramid where we will exploit hexahedral-type symmetries. This reference infinite pyramid is introduced in Section 1.1, and we present the approximation spaces $\mathcal{U}^{(s),k}(\Omega_\infty)$ on this element in Section 2. Our construction of the finite element spaces $\mathcal{U}^{(s),k}(\Omega)$ is detailed in Section 3. We verify that our approximation spaces include the first order elements of [12] in Section 4. We collect shape functions associated with the faces and edges on Ω_∞ in

the Appendix.

Our starting point is an observation: given polynomial data on the faces of a pyramid, it is not immediately obvious if these can be extended in a polynomial-preserving sense to the pyramid. Other authors [11, 20, 24, 16] have proposed using a macro-element that uses piecewise polynomial functions by dividing the pyramid into two or four tetrahedra. The difficulty in finding bases composed of functions that are polynomial across the whole pyramid is not a coincidence. We show in Theorem 1.1 that it is impossible to construct useful $H^1(\Omega)$ pyramidal finite elements using solely polynomial basis functions. We do this by demonstrating an $H^1(\Omega)$ function which has polynomial traces on the faces of the pyramid, but which does not admit a polynomial interpolant within the pyramid.

THEOREM 1.1. *Let Ω be the pyramid defined in (1.1). Consider the function $u : \Omega \rightarrow \mathbb{R}$ defined by*

$$u(\xi, \eta, \zeta) = \frac{\xi\zeta(\xi + \zeta - 1)(\eta + \zeta - 1)}{1 - \zeta}.$$

Then,

1. $u \in H^1(\Omega)$,
2. u has polynomial traces on the pyramid faces,
3. There exists a k_0 such that for all $k \geq k_0$, any finite element space $\mathcal{U}^{(s),k}(\Omega)$, satisfying property **(P1)** does not contain a polynomial function that interpolates u .

Proof. It is straightforward to verify (1). It is easy to see $u|_{\eta=0} = -\xi\zeta(\xi + \zeta - 1)$ and $u = 0$ on the other faces of the pyramid. This establishes (2).

We prove (3) by contradiction. Since Ω has Lipschitz boundary, we can extend u to a function $U \in H^1(\mathbb{R}^3)$ (see, for example, [1]). Choose k_0 so that the corresponding tetrahedral element interpolates $U|_{\eta=0} = -\xi\zeta(\xi + \zeta - 1)$ exactly¹. Suppose that we could represent $u = U|_{\Omega}$ by a polynomial function p , in a manner consistent with property **(P1)**. Since an H^1 -conforming approximation must be continuous and traces of U on the faces will be interpolated exactly by the neighbouring tetrahedra and hexahedra, we must have $p = U$ on each face of the pyramid.

Since $U = u = 0$ on four of the faces of the pyramid, we can factorise:

$$p(\xi, \eta, \zeta) = \xi\zeta(\xi + \zeta - 1)(\eta + \zeta - 1)[r(\xi, \zeta) + \eta s(\xi, \eta, \zeta)], \quad (1.3)$$

where r and s are polynomial. Further, $U = -\xi\zeta(\xi + \zeta - 1)$ on the face $\eta = 0$ and so:

$$p(\xi, 0, \zeta) = \xi\zeta(\xi + \zeta - 1)(\zeta - 1)r(\xi, \zeta) = -\xi\zeta(\xi + \zeta - 1), \quad (1.4)$$

which implies that $(\zeta - 1)r(\xi, \zeta) = -1$. This contradicts the polynomial nature of r . \square

A similar result is presented in [24], where it is claimed that, under the assumption that shape functions must be polynomial, there exists no continuously differentiable conforming shape functions for the pyramid which are linear / bilinear on the faces.

1.1. The infinite reference element: pullbacks. To construct the finite elements, we shall make use of two reference elements: the *finite pyramid*, Ω , already introduced in (1.1), and the *infinite pyramid* Ω_∞ . We will typically use the symbols

¹With the conventional understanding that the k th order continuous element contains all polynomials of degree k , this means let $k_0 = 3$. Note that it is in fact also possible to show that we need rational functions for $k = 1, 2$

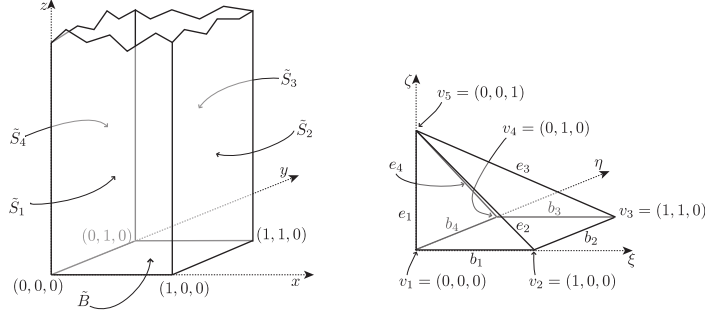


FIG. 1.1. *Left: The infinite pyramid Ω_∞ . Right: The finite reference pyramid Ω*

(x, y, z) as coordinates on the infinite pyramid and (ξ, η, ζ) on the finite pyramid. The infinite reference pyramid is defined as

$$\Omega_\infty = \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 \cup \infty \mid x, y, z \geq 0, x \leq 1, y \leq 1\}. \quad (1.5)$$

To associate the finite and infinite pyramids, define the bijection $\phi : \Omega_\infty \rightarrow \Omega$

$$\phi(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z}, \frac{z}{1+z} \right), \quad \phi(\infty) = (0, 0, 1), \quad (1.6)$$

which is a diffeomorphism if we restrict the domain to $\Omega_\infty \setminus \infty$ (and the range to the finite pyramid with its tip removed).

Figure 1.1 shows the two pyramids. The vertical faces of the infinite pyramid lie in the planes $y = 0$, $x = 1$, $y = 1$, $x = 0$. We denote them as S_{1,Ω_∞} , S_{2,Ω_∞} , S_{3,Ω_∞} , and S_{4,Ω_∞} respectively, and the corresponding faces on the finite pyramid $S_{i,\Omega} = \phi(S_{i,\Omega_\infty})$. Let B_{Ω_∞} refer to the base face, $z = 0$, of the infinite pyramid and B_Ω the base face of the finite pyramid. The vertices of the finite pyramid are denoted v_i , $i = 1..5$, with v_5 the point $(0, 0, 1)$. Denote the base edges of the finite pyramid $b_i = S_{i,\Omega} \cap B_\Omega$ and the other edges $e_i = \overline{v_i v_5}$. Define E as the set of all the edges of the finite pyramid.

The infinite pyramid will serve as a tool for the construction of the function spaces for the elements. We thus need to understand how to map functions between spaces on the finite pyramid, $\mathcal{U}^{(s),k}(\Omega)$ and the infinite pyramid, $\mathcal{U}^{(s),k}(\Omega_\infty)$. We will construct the approximation spaces on the infinite pyramid to satisfy an exact sequence property. To have this exact sequence property preserved on the finite pyramid, it is necessary that the mappings between the spaces on the finite and infinite pyramids commute with the grad, curl and div operators.

In the language of differential geometry, where the elements of each space can be considered to be proxies for 0, 1, 2 and 3-forms, the mappings should be pullbacks. We shall use the same notation for each pullback map - the context will never be ambiguous. We point the reader to [3] for an excellent treatment of the finite element exterior calculus. In this paper, we will switch between referring to objects as forms or functions, depending on the context. Formally (because we have not yet defined

the appropriate Sobolev spaces on the infinite pyramid):

$$\forall u \in H^1(\Omega) \quad \phi^* u = u \circ \phi, \quad (1.7a)$$

$$\forall E \in H(\text{curl}, \Omega) \quad \phi^* E = D\phi^T \cdot [E \circ \phi], \quad (1.7b)$$

$$\forall v \in H(\text{div}, \Omega) \quad \phi^* v = |D\phi| D\phi^{-1} \cdot [v \circ \phi], \quad (1.7c)$$

$$\forall q \in L^2(\Omega) \quad \phi^* q = |D\phi| [q \circ \phi], \quad (1.7d)$$

where $D\phi$ is the Jacobian matrix, $\frac{1}{(z+1)^2} \begin{pmatrix} z+1 & 0 & -x \\ 0 & z+1 & -y \\ 0 & 0 & 1 \end{pmatrix}$. The pullback is a

bijection and the inverse pullback, $(\phi^*)^{-1}$ is equal to $(\phi^{-1})^*$. Since $z \geq 0$, $D\phi^T D\phi$ is positive definite.

1.2. The infinite reference element: Sobolev spaces. The infinite reference pyramid has obvious symmetries, which make it easier to specify and analyze approximation spaces. However, it has semi-infinite extent along the z -direction, and we must therefore describe analogues of $H^1(\Omega)$, $H(\text{curl}, \Omega)$ etc. on Ω_∞ . Not surprisingly, these Sobolev spaces will have weighted norms.

DEFINITION 1.2. *Let Ω_∞ be the infinite pyramid defined in (1.5), and $\phi : \Omega_\infty \rightarrow \Omega$ be the pullback map. We define the inner product spaces*

- $H_w^1(\Omega_\infty)$ is the set of scalar-valued functions $v : \Omega_\infty \rightarrow \mathbb{R}$ with inner product

$$(u, v)_{H_w^1(\Omega_\infty)} := \int_{\Omega_\infty} \frac{uv}{(1+z)^4} + (\nabla u)^T \mathcal{A} \nabla v d\mathbf{x}.$$

Here $\mathcal{A} = |D\phi| D\phi^{-1} D\phi^{-1T}$ is positive definite.

- $H_w(\text{curl}, \Omega_\infty)$ is the set of vector-valued functions (1-forms) $F : \Omega_\infty \rightarrow (\mathbb{R})^3$ with inner product

$$(F, G)_{H_w(\text{curl}, \Omega_\infty)} := \int_{\Omega_\infty} (F)^T \mathcal{A}(G) + (\text{curl } F)^T \mathcal{B}(\text{curl } G) d\mathbf{x}.$$

Here $\mathcal{B} = |D\phi^{-1}| D\phi^T D\phi$, and is positive definite.

- $H_w(\text{div}, \Omega_\infty)$ is the set of vector-valued functions (2-forms) $F : \Omega_\infty \rightarrow (\mathbb{R})^3$ with inner product

$$(F, G)_{H_w(\text{div}, \Omega_\infty)} := \int_{\Omega_\infty} (F)^T \mathcal{B}(G) + (\text{div } F)^T (1+z)^4 (\text{div } G) d\mathbf{x}.$$

- $L_w^2(\Omega_\infty)$ is the set of scalar-valued functions (3-forms) with inner product,

$$(u, v)_{L_w^2(\Omega_\infty)} := \int_{\Omega_\infty} (1+z)^4 (uv) d\mathbf{x}.$$

We observe that the inner products in these Sobolev spaces on the infinite pyramid are weighted by powers of $\frac{1}{(1+z)}$. The subscript w is to make clear that these are weighted norms. It is clear that these inner product spaces are, in fact, Hilbert spaces. We can now identify these Hilbert spaces with the usual Sobolev spaces on the finite pyramid.

LEMMA 1.3. *The spaces $H_w^1(\Omega_\infty)$, $H_w(\text{curl}, \Omega_\infty)$, $H_w(\text{div}, \Omega_\infty)$ and $L_w^2(\Omega_\infty)$ defined above are Hilbert spaces. Moreover, $\phi^* : H^1(\Omega) \rightarrow H_w^1(\Omega_\infty)$ is an isometry. The analogous statements are true for $H_w(\text{curl}, \Omega_\infty)$, $H_w(\text{div}, \Omega_\infty)$ and $L_w^2(\Omega_\infty)$.*

Proof. The pullbacks, ϕ^* are formally bijections because Ω and Ω_∞ have the same dimension. Suppose \tilde{u} is a 0-form in $H^1(\Omega)$ and let $u = \phi^*\tilde{u}$. Then

$$\|\tilde{u}\|_{L^2(\Omega)}^2 = \int_{\Omega_\infty} |D\phi||u(\mathbf{x})|^2 d\mathbf{x} = \int_{\Omega_\infty} \frac{1}{(1+z)^4} |u(\mathbf{x})|^2 d\mathbf{x}.$$

Now, the gradient and pull-back operator commute. Since $\nabla\tilde{u}$ is a one-form, we use the appropriate pull-back to obtain

$$\begin{aligned} \|\nabla\tilde{u}\|_{L^2(\Omega)}^2 &= \int_{\Omega} |\nabla\tilde{u}|^2 d\xi = \int_{\Omega} |D\phi^{-1T} \nabla u \circ \phi^{-1}|^2 d\xi \\ &= \int_{\Omega_\infty} |D\phi| |D\phi^{-1T} \nabla u|^2 d\mathbf{x} = \int_{\Omega_\infty} \nabla u^T \mathcal{A} \nabla u d\mathbf{x}. \end{aligned}$$

Hence $\|\tilde{u}\|_{H^1(\Omega)}^2 = \|\tilde{u}\|_{L^2(\Omega)}^2 + \|\nabla\tilde{u}\|_{L^2(\Omega)}^2 = \|u\|_{H_w^1(\Omega_\infty)}^2$. The proofs for $H_w(\text{curl}, \Omega_\infty)$, $H_w(\text{div}, \Omega_\infty)$ and $L_w^2(\Omega_\infty)$ follow analogously. \square

The weights are entirely specified by the projective mapping, ϕ , and the associated pull-backs for the various forms. It is important to note, for example, that $\|u\|_{L_w^2(\Omega_\infty)}^2 = \int_{\Omega_\infty} \frac{u^2}{(1+z)^4} d\mathbf{x}$ if u is a zero form, while $\|u\|_{L_w^2(\Omega_\infty)}^2 = \int_{\Omega_\infty} u^2 (1+z)^4 d\mathbf{x}$ if u is a 3-form.

We also collect here concrete instantiations of the inverse pullback mapping.

$$\forall u \in H_w^1(\Omega_\infty), \quad (\phi^{-1})^* u = u \circ \phi^{-1}, \quad (1.8a)$$

$$\forall E \in H_w \text{curl}, \Omega_\infty, \quad (\phi^{-1})^* E = [(1+z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1+z \end{pmatrix} \cdot E] \circ \phi^{-1}, \quad (1.8b)$$

$$\forall v \in H_w(\text{div}, \Omega_\infty), \quad (\phi^{-1})^* v = [(1+z)^2 \begin{pmatrix} 1+z & 0 & -x \\ 0 & 1+z & -y \\ 0 & 0 & 1 \end{pmatrix} \cdot v] \circ \phi^{-1}, \quad (1.8c)$$

$$\forall q \in L_w^2(\Omega_\infty), \quad (\phi^{-1})^* q = [(1+z)^4 q] \circ \phi^{-1}. \quad (1.8d)$$

1.3. Rotations and traces. Define $R_{\Omega_\infty} : \Omega_\infty \rightarrow \Omega_\infty$ to be the affine mapping that sends the infinite pyramid to itself and rotates it a quarter turn about the axis $x = y = \frac{1}{2}$, that is, the vertical face S_{1,Ω_∞} is mapped to S_{2,Ω_∞} , the face S_{2,Ω_∞} is mapped to S_{3,Ω_∞} , etc. Explicitly,

$$R_{\Omega_\infty} : (x, y, z) \mapsto (1-y, x, z). \quad (1.9)$$

We can also define a mapping that sends the finite pyramid to itself, rotating the faces, $R : \Omega \rightarrow \Omega$ by

$$R = \phi \circ R_{\Omega_\infty} \circ \phi^{-1}, \quad R : (\xi, \eta, \zeta) \mapsto (1-\eta-\zeta, \xi, \zeta).$$

It is clear that if an approximation space $\mathcal{U}^{(s),k}(\Omega_\infty)$ is invariant under the mapping R_{Ω_∞} , its (inverse) pullback to the finite pyramid will be invariant under R . This property will prove convenient when we consider exterior shape functions and exterior degrees of freedom.

From differential geometry, we expect that zero trace data will be preserved by the pullback mapping. The following lemma makes this explicit in our concrete vector calculus formulation. We suppose that S_{Ω_∞} is a surface of the infinite pyramid and let S_Ω be its image under ϕ on the finite pyramid (although the lemma is true in general for any pair of surfaces, S_1 and S_2 and bijection $\psi : S_1 \rightarrow S_2$).

LEMMA 1.4.

- A 1-form u is normal to S_Ω at a point $\xi = \phi(x)$ if and only if the pullback ϕ^*u is normal to S_{Ω_∞} at x .
- A 2-form u is tangent to S_Ω at a point $\xi = \phi(x)$ if and only if the pullback ϕ^*u is tangent to S_{Ω_∞} at x .

Proof. Let S_Ω be described (locally) by $S_\Omega = \{\xi : f(\xi) = 0\}$. Define $g = f \circ \phi$, then $S_{\Omega_\infty} = \{x : g(x) = 0\}$. To establish the first result, let u be a 1-form which is normal to S_Ω at ξ , then

$$u(\xi) = \lambda(\xi) \nabla f(\xi) \quad (1.10)$$

for some scalar function λ . By the chain rule, and substituting (1.10)

$$\begin{aligned} \nabla g(x) &= (D\phi)^T(x) \cdot (\nabla f)(\phi(x)) = (D\phi)^T(x) \cdot \frac{u(\phi(x))}{\lambda(\phi(x))} = \frac{\phi^*u}{\lambda(\phi(x))} \\ &\Rightarrow \lambda(\phi(x)) \nabla g(x) = \phi^*u(x). \end{aligned}$$

Hence, ϕ^*u is normal to S at x . Since ϕ is a bijection, the converse is also true. To establish the second result, let u be a 2-form which is tangent to S_Ω then $u \cdot \nabla f = 0$. The chain rule gives us $\nabla g = (D\phi)^T(\nabla f) \circ \phi$ and by definition of the pullback, $\phi^*u = |D\phi|(D\phi)^{-1} \cdot (u \circ \phi)$, hence:

$$\begin{aligned} \phi^*u \cdot \nabla g &= |D\phi|(u \circ \phi)^T \cdot (D\phi^{-1})^T \cdot (D\phi)^T \cdot [(\nabla f) \circ \phi] \\ &= |D\phi|(u^T \cdot \nabla f) \circ \phi = 0. \end{aligned}$$

Hence ϕ^*u is tangent to S_{Ω_∞} . Since ϕ is a bijection, the converse is also true. \square

In order to satisfy property **P1** we need to understand the traces of the functions in the approximation spaces of the tetrahedral and hexahedral elements to which the pyramid will be patched. We will use the following notation for trace maps:

DEFINITION 1.5. For $s = 0, 1, 2$ define:

- $\Gamma_{i,\Omega_\infty}^s$: the trace to each vertical face, S_{i,Ω_∞} of the infinite pyramid, Ω_∞ for $i = 0, 1, 2, 3$.
- $\Gamma_{i,\Omega}^s$: the trace to each face $S_{i,\Omega}$ of the finite pyramid, Ω for $i = 0, 1, 2, 3$.
- $\Gamma_{B,\Omega_\infty}^s$ and $\Gamma_{B,\Omega}^s$: the traces to B_{Ω_∞} and B_Ω .

When $s = 0$, the trace of a suitably smooth function is its point value on the boundary; when $s = 1$, it is the tangential components and when $s = 2$ it is the normal component. It is helpful to recognise that if u is a differential form defined on a manifold M , its trace map, Γ_N on some sub-manifold, N , is the pullback of the inclusion map, $N \hookrightarrow M$. We refer the reader to ([2], pg 41 ff.) for a more detailed exposition. The consequence for us is that trace maps commute with ϕ^* , (e.g. $\Gamma_{i,\Omega_\infty}^s \circ \phi^* = \phi^* \circ \Gamma_{i,\Omega}^s$) so results we establish on faces and edges of Ω_∞ will carry over to the finite pyramid.

Using [17] as our source, we catalogue here the traces of the tetrahedral and hexahedral approximation spaces that must be matched by our spaces, $\mathcal{U}^{(s),k}(\Omega)$. Since our construction will start on the infinite pyramid, we also show some of the pullbacks of these spaces to the infinite pyramid.

- *Compatibility of traces* $\Gamma_{i,\Omega_\infty}^0 u$: The approximation space for the tetrahedral H^1 -conforming element is P^k , therefore if $u \in \mathcal{U}^{(0),k}(\Omega)$, it must satisfy $\Gamma_{1,\Omega}^0 u \in P^k[\xi, \zeta]$. That is, $\Gamma_{1,\Omega}^0 u$ is a sum of terms of the form $\xi^a \zeta^c (1-\zeta)^{k-a-c}$ with $a + c \leq k$. Pulled back to the infinite pyramid, we see that these are

of the form $\Gamma_{1,\Omega_\infty}^0 \phi^* u = \frac{x^a z^c}{(1+z)^k} \in P_k^k[x, z]$.² By symmetry we need similar conditions on the other vertical faces of the pyramid. The approximation space for the hexahedral element is $Q^{k,k,k}$ and so we need $\Gamma_{B,\Omega}^0 u \in Q^{k,k,k}[x, y]$

- *Compatibility of traces $\Gamma_{i,\Omega}^1 u$:* For 1-forms, the approximation space for the k th order $H(\text{curl})$ -conforming tetrahedral element presented in [17] is $(P^{k-1})^3 \oplus S^{k,3}$, where $S^{k,3} = \{u \in (\tilde{P}^k)^3 \mid (\xi - \xi_0) \cdot u = 0\}$ for some arbitrary base point, ξ_0 . We can take $\xi_0 = (0, 0, 1)$, and observe that the face trace onto $S_{i,\Omega}$ must be contained in $(P^{k-1})^2 \oplus S^{k,2}$ where $S^{k,2} = \{w \in (\tilde{P}^k)^2 \mid (\xi - \xi_0) \cdot w = 0\}$. This specifies the compatibility constraint the trace $\Gamma_{1,\Omega}^1 u$ must obey. The pullback of this space is $(P_{k+1}^{k-1}[x, z])^2 \oplus \tilde{P}_{k+1}^{k-1}[x, 1+z] \begin{pmatrix} 1+z \\ -x \end{pmatrix}$.

We can use symmetry to define similar conditions on the other vertical faces of the pyramid. The traces of k th order hexahedral conforming elements for $H(\text{curl})$ lie in the polynomial space $Q^{k-1,k}(x, y) \times Q^{k,k-1}(x, y)$.

- *Compatibility of traces $\Gamma_{i,\Omega}^2 u$:* For 2-forms, $\Gamma_{i,\Omega}^2 u$ are the normal traces of u onto the face S_i of the pyramid. The polynomial space used for the $H(\text{div})$ -conforming element on the tetrahedron is $(P^{k-1})^3 \oplus \xi \tilde{P}^{k-1}$. The functions derived from the homogeneous polynomial space make no contribution to the face traces. Therefore, the normal traces onto the faces of a tetrahedron will lie in (P^{k-1}) , and this is the compatibility condition we must ensure for $\Gamma_{i,\Omega}^2$. Consider a 2-form, u , whose normal trace is $u \cdot n \in P^{k-1}$ on face S_1 . The pull-back of the associated normal vector field to the infinite pyramid will have the form

$$\phi^* u = \frac{1}{(1+z)^{k+2}} \begin{pmatrix} 0 \\ x^a z^c (1+z)^{k-1-a-c} \\ 0 \end{pmatrix}, \quad a+c \leq k-1.$$

So, $\Gamma_{1,\Omega_\infty}^2 \phi^* u \in P_{k+2}^{k-1}[x, z]$.

The discussions above suggest the face-wise constraints which must be satisfied by any approximation spaces $\mathcal{U}^s(\Omega)$. However, as was demonstrated by Theorem 1.1 the difficulty of pyramidal interpolation stems from the need to find interpolants that match trace data on all the faces simultaneously. This point will be discussed later.

2. The approximation spaces $\mathcal{U}^{(s),k}(\Omega_\infty)$ on the infinite pyramid. We are now ready to construct the approximation spaces on the infinite pyramid. These will be used, via the pullback map, to construct the approximation spaces on the finite pyramid. As a preliminary step, we identify families of “rational polynomials” on Ω_∞ which will be used extensively. We want the spaces on the finite pyramid Ω to contain all polynomials up to a specified degree. Consider the effect of the pullback mapping ϕ on a polynomial of degree k , $p = \xi^\alpha \eta^\beta \zeta^\gamma \in H^1(\Omega)$, where $\alpha + \beta + \gamma = k$:

$$\phi^* p = \frac{x^\alpha y^\beta z^\gamma}{(1+z)^k}. \quad (2.1)$$

From Lemma 1.3, $\phi^* p \in H_w^1(\Omega_\infty)$. This motivates our next definition:

DEFINITION 2.1. *Let $Q^{l,m,n}(x, y, z)$ to be the space of polynomials of maximum degree l, m, n in x, y, z respectively. Define the space of k -weighted tensor product*

²the space P_k^k is defined in (2.3)

polynomials

$$Q_k^{l,m,n}(x, y, z) = \left\{ \frac{u}{(1+z)^k} : u \in Q^{l,m,n}(x, y, z) \right\}.$$

It will be helpful to remember the inclusion:

$$Q_k^{l,m,n} \subset Q_{k+1}^{l,m,n+1}. \quad (2.2)$$

Let $P^n(x, y, z)$ be polynomials of maximum total degree n in (x, y, z) and define the space of k -weighted polynomials of degree n

$$P_k^n(x, y, z) = \left\{ \frac{u(x, y, z)}{(1+z)^k} : u(x, y, z) \in P^n \right\}. \quad (2.3)$$

2.1. $H_w^1(\Omega_\infty)$ -conforming approximation spaces. We begin our construction by noting that in [17], the finite element approximation space for a hexahedral element consists of polynomials of the form $p = \xi^\alpha \eta^\beta \zeta^\gamma$. From (2.1), we know that $\phi^* p = \frac{x^\alpha y^\beta z^\gamma}{(1+z)^k} \in H^1(\Omega_\infty)$, if $\alpha + \beta + \gamma = k$. We might therefore expect to base an approximation space for $H^1(\Omega_\infty)$ on the k -weighted space, $Q_k^{k,k,k}$. However, there are some elements of $Q_k^{k,k,k}$ which, when pulled back to the finite pyramid, become undefined at $\xi_0 = (0, 0, 1)$. The problem arises with elements of the form $\frac{x^\alpha y^\beta z^k}{(1+z)^k}$ on the infinite pyramid. The following examples are illustrative.

EXAMPLE 2.2. Consider the monomial $p_1(x, y, z) = x$ on the infinite pyramid. The inverse pull-back onto the finite pyramid is $(\phi^{-1})^* p = \frac{\xi}{1-\zeta}$. The limit $\lim_{\xi \rightarrow \xi_0} (\phi^{-1})^* p$ depends on the path by which we approach ξ_0 . Specifically, if we take the path $\alpha_\lambda(t) = (\lambda(1-t), 0, t)$ then $\lim_{t \rightarrow 1} (\phi^{-1})^* p(\alpha_\lambda(t)) = \lambda$.

EXAMPLE 2.3. Consider the function $p_2(x, y, z) = \frac{z^k}{(1+z)^k}$ on the infinite pyramid. Pulled back to the finite pyramid, $(\phi^{-1})^* p_2 = \zeta^k$. We must therefore retain p_2 in the approximation space on the infinite pyramid.

LEMMA 2.4. Let Ω_∞ be the infinite pyramid described above, and $k \geq 1$ be a fixed integer.

- Functions $p(x, y, z) := \frac{x^a y^b z^c}{(1+z)^k} \in Q_k^{k,k,k-1}$ satisfy $p \in H_w^1(\Omega_\infty)$.
- If $p(x, y, z) = \frac{r(x, y) z^k}{(1+z)^k}$, $r(x, y) \in Q^{k,k}(x, y)$, then $\lim_{\xi \rightarrow \xi_0} (\phi^{-1})^* (p)$ is only well-defined if $r(x, y) \equiv 1$.

Proof. We can verify the first statement by using Definition 1.2. The second statement can be proved by contradiction, as in Example 2.2. \square

This result and the examples suggest the basis functions to include in a finite-dimensional approximation space for $H_w^1(\Omega_\infty)$.

DEFINITION 2.5. Let k be a positive integer. We define the underlying spaces:

$$\overline{\mathcal{U}^{(0),k}}(\Omega_\infty) = Q_k^{k,k,k-1} \oplus \text{span} \left\{ \frac{z^k}{(1+z)^k} \right\}. \quad (2.4)$$

LEMMA 2.6. The rational polynomials $\left\{ \frac{x^a y^b z^c}{(1+z)^k}, 0 \leq a, b \leq k, 0 \leq c \leq k-1 \right\}$ and $\frac{z^k}{(1+z)^k}$ form a basis for $\overline{\mathcal{U}^{(0),k}}(\Omega_\infty)$. Moreover, $\overline{\mathcal{U}^{(0),k}}(\Omega_\infty)$ can be represented as

$$\overline{\mathcal{U}^{(0),k}}(\Omega_\infty) = \{u \in Q_k^{k,k,k} : \nabla u \in Q_k^{k-1,k,k-1} \times Q_k^{k,k-1,k-1} \times Q_{k+1}^{k,k,k-1}\}. \quad (2.5)$$

Proof. The basis functions are determined by using the definition of $\overline{\mathcal{U}^{(0),k}}(\Omega_\infty)$ and Lemma 2.4. The gradients of rational functions of the form $\frac{x^a y^b z^c}{(1+z)^k}$ are 1-forms in $Q_k^{k-1,k,k-1} \times Q_k^{k,k-1,k-1} \times Q_{k+1}^{k,k,k-1}$. Moreover, $\nabla \frac{z^k}{(1+z)^k} = (0, 0, \frac{kz^{k-1}}{(1+z)^{k+1}})^T$. The reverse inclusion follows readily by a similar calculation. This establishes the alternative characterization of $\mathcal{U}^{(0),k}(\Omega_\infty)$. \square

We must now constrain these spaces to obtain the approximation spaces which satisfy the compatibility constraints **P1**. This follows the discussion in Section 1.3.

DEFINITION 2.7. *Let k be a positive integer. We define the k -th order approximation spaces $\mathcal{U}^{(0),k}(\Omega_\infty)$:*

$$\mathcal{U}^{(0),k}(\Omega_\infty) = \{u \in \overline{\mathcal{U}^{(0),k}}(\Omega_\infty) \mid \Gamma_{1,\Omega_\infty} \in P_k^k[x, z], \text{ similarly on } S_{i,\Omega_\infty}, i = 2, 3, 4\}. \quad (2.6)$$

Since we will be working in the projection-based interpolation framework while specifying internal degrees of freedom, we define a subspace $\mathcal{U}_0^{(0),k}(\Omega_\infty)$, consisting of functions in $\mathcal{U}^{(0),k}(\Omega_\infty)$ with zero trace on the boundary of Ω_∞ . Clearly, $\mathcal{U}_0^{(0),k}(\Omega_\infty) = \{x(1-x)y(1-y)zu, u \in Q_k^{k-2,k-2,k-2}\}$.

In the Appendix, we present the shape functions in $\mathcal{U}^{(0),k}(\Omega_\infty)$ associated with the faces, edges and vertices of Ω_∞ . These are linearly independent. Moreover, the number of these functions associated with a given face is exactly the same as the dimension of trace spaces from neighbouring simplices.

2.2. $H_w(\text{curl}, \Omega_\infty)$ -conforming approximation spaces. We now present the construction of the approximation space $\mathcal{U}^{(1),k}(\Omega_\infty)$ of $H_w(\text{curl}, \Omega_\infty)$. As before, this construction is motivated by the ultimate goal of constructing a finite element approximation space for $H(\text{curl}, \Omega)$ which satisfies property **(P1)**.

To satisfy the commuting diagram property we will need, at the very least, to have $\nabla \mathcal{U}^{(0),k}(\Omega_\infty) \subset \mathcal{U}^{(1),k}(\Omega_\infty)$. The alternate characterization of $\overline{\mathcal{U}^{(0),k}}(\Omega_\infty)$ in Lemma 2.6 suggests that we might consider the space $Q_k^{k-1,k,k-1} \times Q_k^{k,k-1,k-1} \times Q_{k+1}^{k,k,k-1}$ as a candidate for an approximation space for $H(\text{curl}, \Omega_\infty)$. However, this space includes functions that are undefined at the point $\xi_0 = (0, 0, 1)$ on the finite pyramid. We must be careful here to identify what kind of discontinuities we wish to exclude on the finite pyramid. We are not interested in point values of our functions, only the tangential components.

EXAMPLE 2.8. Consider $u = \begin{pmatrix} y/(1+z) \\ 0 \\ 0 \end{pmatrix} \in Q_k^{k-1,k,k-1} \times Q_k^{k,k-1,k-1} \times Q_{k+1}^{k,k,k-1}$.

Its (inverse) pullback to the finite pyramid is, $(\phi^{-1})^*u = \begin{pmatrix} \eta/(1-\zeta) \\ 0 \\ \xi\eta/(1-\zeta)^2 \end{pmatrix}$.

Let $\bar{v} = (0, -1, 1)$ and consider the path $\alpha_\lambda(t) = (\lambda(1-t), 1-t, t)$. This path lies on the face S_3 for $\lambda \in [0, 1]$, and S_3 is tangent to \bar{v} . The limit of the component of $(\phi^{-1})^*u$ tangent to \bar{v} at ξ_0 along the path α_λ is $\lim_{t \rightarrow 1} u(\alpha_\lambda(t)) \cdot \bar{v} = \lambda$.

EXAMPLE 2.9. Let $u = \frac{z^{k-1}}{(1+z)^{k+1}} \begin{pmatrix} r_x z \\ r_y z \\ -r \end{pmatrix}$, $r \in Q^{k,k}[x, y]$, be a 1-form defined on

the infinite pyramid. Note that we can write $u = \nabla(\frac{rz^k}{(1+z)^{k+1}}) - \begin{pmatrix} 0 \\ 0 \\ \frac{(k+1)rz^{k-1}}{(1+z)^{k+2}} \end{pmatrix}$, from which it is apparent that $u \in H_w(\text{curl}, \Omega_\infty)$.

With these examples in hand, we are able to define approximation spaces for $H_w(\text{curl}, \Omega_\infty)$.

DEFINITION 2.10. *Let $k \geq 1$ be an integer. We define the underlying space*

$$\overline{\mathcal{U}}^{(1),k}(\Omega_\infty) := Q_{k+1}^{k-1,k,k-1} \times Q_{k+1}^{k,k-1,k-1} \times Q_{k+1}^{k,k,k-2} \oplus \left\{ \frac{z^{k-1}}{(1+z)^{k+1}} \begin{pmatrix} r_x z \\ r_y z \\ -r \end{pmatrix}, \quad r \in Q^{k,k}[x, y] \right\}. \quad (2.7)$$

An equivalent characterization of the underlying space $\overline{\mathcal{U}}^{(1),k}(\Omega_\infty)$ is given as

$$\overline{\mathcal{U}}^{(1),k}(\Omega_\infty) = \{u \in Q_{k+1}^{k-1,k,k} \times Q_{k+1}^{k,k-1,k} \times Q_{k+1}^{k,k,k-1} : \nabla \times u \in Q_{k+2}^{k,k-1,k-1} \times Q_{k+2}^{k-1,k,k-1} \times Q_{k+2}^{k-1,k-1,k}\}, \quad (2.8)$$

We add surface constraints to get the full definition of the approximation space: let n_i be the (outward) normal to the vertical faces S_{i,Ω_∞} of Ω_∞ . Then $\Gamma_{i,\Omega_\infty}^1 u := u \times n_i|_{S_{i,\Omega_\infty}}$.

DEFINITION 2.11. *Let $k \geq 1$ be an integer. Define*

$$\mathcal{U}^{(1),k}(\Omega_\infty) = \left\{ u \in \overline{\mathcal{U}}^{(1),k} \mid \Gamma_{1,\Omega_\infty}^1 u \in (P_{k+1}^{k-1}[x, z])^2 \oplus \tilde{P}_{k+1}^{k-1}[x, 1+z] \begin{pmatrix} 1+z \\ -x \end{pmatrix} \right. \\ \left. \text{and similarly on } S_{i,\Omega_\infty}, \quad i = 2, 3, 4, \right\}, \quad (2.9)$$

where

$$\tilde{P}_{k+1}^{k-1}[x, 1+z] = \frac{1}{(1+z)^{(k+1)}} \text{span} \{x^a(1+z)^{k-1-a}, \quad 0 \leq a \leq k-1\}.$$

We can denote the subset $\mathcal{U}^{(1),k}(\Omega_\infty)$ consisting of elements whose (tangential) traces vanish on $\partial\Omega_\infty$ as $\mathcal{U}_0^{(1),k}(\Omega_\infty)$.

As for $\mathcal{U}^{(0),k}(\Omega_\infty)$, in the Appendix we have tabulated the edge and face shape functions for Ω_∞ . These are linearly independent, and are consistent along shared edges. The same will be true of the pull-backs onto the finite pyramid.

2.3. $H_w(\text{div}, \Omega_\infty)$ and $L_w^2(\Omega_\infty)$ -conforming approximation spaces. Following a similar strategy to the previous sections, we wish to construct approximation spaces $\mathcal{U}^{(2),k}(\Omega_\infty)$ for $H_w(\text{div}, \Omega_\infty)$, such that their pull-backs to the finite pyramid provide approximation spaces for $H(\text{div}, \Omega)$. Again, we want $\text{curl } u \in \mathcal{U}^{(2),k}(\Omega_\infty)$, $\forall u \in \overline{\mathcal{U}}^{(1),k}(\Omega_\infty)$. Now, the curls of functions $u \in \overline{\mathcal{U}}^{(1),k}(\Omega_\infty)$ satisfy

$$\nabla \times u \in Q_{k+2}^{k,k-1,k-1} \times Q_{k+2}^{k-1,k,k-1} \times Q_{k+1}^{k-1,k-1,k-1}.$$

Not all of these will have well-defined normal traces, and we must exclude these.

DEFINITION 2.12. *The underlying space for the $H(\text{div})$ -conforming element is defined as:*

$$\begin{aligned} \overline{\mathcal{U}^{(2),k}}(\Omega_\infty) &= Q_{k+2}^{k,k-1,k-2} \times Q_{k+2}^{k-1,k,k-2} \times Q_{k+2}^{k-1,k-1,k-1} \\ &\oplus \frac{z^{k-1}}{(1+z)^{k+2}} \begin{pmatrix} 0 \\ 2s \\ s_y(1+z) \end{pmatrix} \oplus \frac{z^{k-1}}{(1+z)^{k+2}} \begin{pmatrix} 2t \\ 0 \\ t_x(1+z) \end{pmatrix}. \end{aligned} \quad (2.10)$$

Here $s(x, y) \in Q^{k-1,k}[x, y]$, $t(x, y) \in Q^{k,k-1}[x, y]$. An alternate characterization of this space is

$$\overline{\mathcal{U}^{(2),k}}(\Omega_\infty) = \{u \in Q_{k+2}^{k,k-1,k-1} \times Q_{k+2}^{k-1,k,k-1} \times Q_{k+2}^{k-1,k-1,k} : \nabla \cdot u \in Q_{k+3}^{k-1,k-1,k-1}\}. \quad (2.11)$$

We equip this space with surface constraints to obtain the full definition of the approximation space on the infinite pyramid:

DEFINITION 2.13. *The k th order approximation space for $H_w(\text{div}, \Omega_\infty)$ is*

$$\mathcal{U}^{(2),k}(\Omega_\infty) = \{u \in \overline{\mathcal{U}^{(2),k}} \mid \Gamma_{1,\Omega_\infty}^{(2)} \in P_{k+2}^{k-1}[x, z] \text{ similarly on } S_{i,\Omega_\infty}, i = 2, 3, 4\}. \quad (2.12)$$

Again, we can identify the 2-forms in $\mathcal{U}^{(2),k}(\Omega_\infty)$ with vanishing normal traces on the faces of Ω_∞ . We denote this set by $\mathcal{U}_0^{(2),k}(\Omega_\infty)$. In the Appendix, we have written down a basis for $\mathcal{U}_0^{(2),k}(\Omega_\infty)$, and augmented it with shape functions for the faces.

Since we want the divergence operator to be surjective as a map from $\mathcal{U}^{(2),k}(\Omega_\infty)$ to the associated approximation space of $L_w^2(\Omega_\infty)$, the approximation space for $L^2(\Omega_\infty)$ (considered as the space of 3-forms) consists of precisely $\text{div } \mathcal{U}^{(2),k}(\Omega_\infty)$. There is no longer any need to define an underlying space.

DEFINITION 2.14. *We define the approximation space $\mathcal{U}^{(3),k}(\Omega_\infty)$ for $L_w^2(\Omega_\infty)$ as*

$$\mathcal{U}^{(3),k}(\Omega_\infty) = Q_{k+3}^{k-1,k-1,k-1}. \quad (2.13)$$

3. The approximation spaces $\mathcal{U}^{(s),k}(\Omega)$ on the finite pyramid. We are now readily able to demonstrate the approximation spaces for the de Rham sequence on the finite pyramid. The construction is based on the approximation spaces on the infinite pyramid Ω_∞ :

DEFINITION 3.1. *Let Ω be the finite reference pyramid as defined in (1.1). Then, the k th order conforming subspaces on the finite pyramid Ω are*

$$\mathcal{U}^{(s),k}(\Omega) := \left\{ (\phi^{-1})^* u : u \in \mathcal{U}^{(s),k}(\Omega_\infty) \right\}, \quad s = 0, 1, 2, 3. \quad (3.1)$$

THEOREM 3.2. *Let k be a positive integer. The finite dimensional spaces defined in (3.1) satisfy:*

$$\mathcal{U}^{(0),k}(\Omega) \subset H^1(\Omega), \quad \mathcal{U}^{(1),k}(\Omega) \subset H(\text{curl}, \Omega), \quad (3.2)$$

$$\mathcal{U}^{(2),k}(\Omega) \subset H(\text{div}, \Omega), \quad \mathcal{U}^{(3),k}(\Omega) \subset L^2(\Omega). \quad (3.3)$$

Proof. The proof follows from the definitions and properties of $\mathcal{U}^{(s),k}(\Omega_\infty)$, the pull-back map ϕ , and Lemma 1.3. \square

In the following subsections, we shall establish several useful properties of these spaces. The analysis will typically be performed for the approximation spaces on the infinite pyramid, where the basis functions are tensorial in nature, and hexahedral symmetries can be used, which allows for simple calculations in many cases. The properties of the pull-back operator will allow us to demonstrate the results on the finite pyramid.

3.1. $H^1(\Omega)$ -conforming approximation spaces. In this section, we demonstrate that the grad operator is injective on $\mathcal{U}_0^{(0),k}(\Omega)$, the set of bubble functions on the pyramid.

LEMMA 3.3. *Let $\mathcal{U}_0^{(0),k}(\Omega)$ be the subset of $\mathcal{U}^{(0),k}(\Omega)$, consisting of functions whose trace onto the faces and edges of Ω are zero. If $\nabla v = 0$ for some $v \in \mathcal{U}_0^{(0),k}(\Omega)$, $v \equiv 0$ on Ω .*

Proof. This follows from the divergence theorem. \square

We can easily see that $\mathcal{U}_0^{(0),k}(\Omega) = \{(\phi^{-1})^*u : u \in \mathcal{U}_0^{(0),k}(\Omega_\infty)\}$. From the remarks following (2.6), it follows that $\dim \mathcal{U}_0^{(0),k}(\Omega) = \dim \mathcal{U}_0^{(0),k}(\Omega_\infty) = (k-1)^3$. Note that from the definition of $\mathcal{U}_0^{(0),k}(\Omega_\infty)$ and the discussion in Section 1.3, the face traces of functions in $\mathcal{U}^{(0),k}(\Omega)$ are compatible with those of neighbouring tetrahedral and hexahedral elements. Finally, the shape functions in the Appendix show that the edge traces are well-defined, and that edge traces can be specified in consistent manner.

3.2. $H(\text{curl}, \Omega)$ -conforming approximation spaces. We shall establish that the grad operator maps $\mathcal{U}^{(0),k}(\Omega)$ into $\mathcal{U}^{(1),k}(\Omega)$. This is an important step towards showing exactness of the diagram in 1.2. We then show that the curl operator is injective on a certain subspace of $\mathcal{U}^{(1),k}(\Omega)$, which will be used in establishing unisolvency of the edge elements on the pyramid. We will finally demonstrate a discrete Helmholtz decomposition. Note that from the definition of $\mathcal{U}_0^{(1),k}(\Omega_\infty)$ and the discussion in Section 1.3, the face traces of functions in $\mathcal{U}^{(1),k}(\Omega)$ are compatible with those of neighbouring tetrahedral and hexahedral elements.

LEMMA 3.4. *The gradient operator is well defined as a map from $\overline{\mathcal{U}^{(0),k}(\Omega)}$ into $\overline{\mathcal{U}^{(1),k}(\Omega)}$.*

Proof. It is easier to work on the infinite pyramid. Recall that a basis for $\overline{\mathcal{U}^{(0),k}(\Omega_\infty)}$ is given by functions of the form $u_{a,b,c} = \frac{x^a y^b z^c}{(1+z)^k}$, where a, b and c are integers and $a \in [0, k]$, $b \in [0, k]$ and $c \in [0, k-1]$ or $u_{a,b,c} = \frac{z^k}{(1+z)^k}$. We will show that the gradients of each of these functions lie in $\overline{\mathcal{U}^{(1),k}(\Omega)}$. The result is trivial for $c = 0$. For $c \geq 1$,

$$\nabla u_{a,b,c} = \frac{1}{(1+z)^{k+1}} \begin{pmatrix} ax^{a-1}y^b(z^{c+1} + z^c) \\ bx^ay^{b-1}(z^{c+1} + z^c) \\ x^ay^b((c-k)z^c + cz^{c-1}) \end{pmatrix}.$$

If $c \leq k-2$ then $\nabla u_{a,b,c} \in Q_{k+1}^{k-1,k,k-1} \times Q_{k+1}^{k,k-1,k-1} \times Q_{k+1}^{k,k,k-2}$. In the case $c = k-1$, we can let $r = x^a y^b$ in (2.7) and then the remainder

$$\nabla u_{a,b,c} - \frac{z^{k-1}}{(1+z)^{k+1}} \begin{pmatrix} r_x z \\ r_y z \\ -r \end{pmatrix} = \frac{1}{(1+z)^{k+1}} \begin{pmatrix} a x^{a-1} y^b z^{k-1} \\ b x^a y^{b-1} z^{k-1} \\ c x^a y^b z^{k-2} \end{pmatrix}, \quad (3.4)$$

which is in $Q_{k+1}^{k-1,k,k-1} \times Q_{k+1}^{k,k-1,k-1} \times Q_{k+1}^{k,k,k-2}$. Finally, if $c = k$ then choosing $r = 1$ in (2.7) suffices. Now use the definition of $\mathcal{U}^{(s),k}(\Omega)$ in terms of the inverse pull-back of functions in $\overline{\mathcal{U}}^{(s),k}(\Omega_\infty)$, and the commutativity of the grad with the pull-backs, to conclude the result. \square

Note that the previous result also follows immediately from the (unproven) equivalent characterisations of the underlying spaces, (2.5) and (2.8).

We need to identify the functions in $\mathcal{U}^{(1),k}(\Omega)$ with vanishing tangential traces.

DEFINITION 3.5. Define $\mathcal{U}_0^{(1),k}(\Omega)$ to be the subspace of functions in $\mathcal{U}^{(1),k}(\Omega)$ with zero tangential component on the boundary of Ω . Define $\mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega) \subset \mathcal{U}_0^{(1),k}(\Omega)$ to be the set of gradient-free functions in $\mathcal{U}_0^{(1),k}(\Omega)$.

From Lemma 1.4, we know that if $u \in \mathcal{U}^{(1),k}(\Omega)$ has zero tangential traces on a particular face or edge of Ω , then its pullback to Ω_∞ will have zero tangential traces on the associated face or edge. This allows us to characterize $\mathcal{U}_0^{(1),k}(\Omega)$ and $\mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega)$.

LEMMA 3.6. Functions in $\mathcal{U}_0^{(1),k}(\Omega)$ can be represented as $(\phi^{-1})^*(u)$, where $u \in \mathcal{U}_0^{(1),k}(\Omega_\infty)$ have the form

$$u = \begin{pmatrix} y(1-y)zq_1 \\ x(1-x)zq_2 \\ x(1-x)y(1-y)q_3 \end{pmatrix} + \frac{z^{k-1}}{(1+z)^{k+1}} \begin{pmatrix} r_x z \\ r_y z \\ -r \end{pmatrix}, \quad (3.5)$$

where $q \in Q_{k+1}^{k-1,k-2,k-2} \times Q_{k+1}^{k-2,k-1,k-2} \times Q_{k+1}^{k-2,k-2,k-2}$ and $r = x(1-x)y(1-y)\rho$, $\rho \in Q^{k-2,k-2}[x, y]$.

Proof. It is easily verified that the functions u above have zero tangential traces on the edges and faces of Ω_∞ , and therefore their inverse pullbacks $(\phi^{-1})^*(u)$ belong to $\mathcal{U}_0^{(1),k}(\Omega)$. Note that it follows that $\dim \mathcal{U}_0^{(1),k}(\Omega) = \dim \mathcal{U}_0^{(1),k}(\Omega_\infty) = k(k-1)^2 + k(k-1)^2 + (k-1)^3 + (k-1)^2 = 3k(k-1)^2$. \square

The curl operator has a non-empty null space in $\mathcal{U}_0^{(1),k}(\Omega)$, consisting of gradients. We can precisely characterize the complement of the gradients in $\mathcal{U}_0^{(1),k}(\Omega)$.

LEMMA 3.7. Let $\mathcal{U}_{0,\text{curl}}^{(1),k} := \{v | v = (\phi^{-1})^*u, u \in \mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega_\infty)\}$ where $\mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega_\infty) \subset \mathcal{U}_0^{(1),k}(\Omega_\infty)$ consists of functions u of the form

$$u = \begin{pmatrix} y(1-y)zq_1 \\ x(1-x)zq_2 \\ x(1-x)y(1-y)\rho \end{pmatrix}, \quad (3.6)$$

where $q_1 \in Q_{k+1}^{k-1,k-2,k-2}$, $q_2 \in Q_{k+1}^{k-2,k-1,k-2}$, $\rho \in Q_{k+1}^{k-2,k-2}[x, y]$. Then $\mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega) \subset \mathcal{U}_0^{(1),k}(\Omega)$, and the curl operator is injective on $\mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega)$. In other words, $\text{grad } \mathcal{U}_0^{(0),k}(\Omega) \cap \mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega) = \{0\}$.

Proof. The set inclusion $\mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega) \subset \mathcal{U}_0^{(1),k}(\Omega)$ follows by the definitions of $\mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega)$ and $\mathcal{U}_0^{(1),k}(\Omega)$. To see that the curl operator is injective on $\mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega)$, we first show that the curl operator is injective on $\mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega_\infty)$. The argument proceeds by contradiction.

If $k = 1$ then $\mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega_\infty)$ is empty. Assume $k \geq 2$ and let $u \in \mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega_\infty)$ be as in (3.6). Let either ρ or q_2 not equal to zero and write $\rho = \frac{r(x,y)}{(1+z)^{k+1}}$, $r \in Q^{k-2,k-2}(x,y)$. Suppose that $\nabla \times u = 0$. From the x -component, we obtain

$$\frac{1}{(1+z)^{k+1}} \frac{\partial}{\partial y} (y(1-y)r) - \frac{\partial}{\partial z} (zq_2) = 0.$$

There is no z -dependence in r so we can factorise $q_2 = f(z)g(x,y)$, where $f \in P^{k-2}(z)$ satisfies

$$\frac{d}{dz} \frac{zf(z)}{(1+z)^{k+1}} = \frac{1}{(1+z)^{k+1}}.$$

This is impossible, and so $\rho = q_2 = 0$. A similar consideration of the y -component shows that $q_1 = 0$. We have just established that the curl operator is injective on $\mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega_\infty)$. Since the pullback and curl commute, the curl is injective on $\mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega)$. \square

We can now state a discrete Helmholtz decomposition for $\mathcal{U}_0^{(1),k}(\Omega)$:

THEOREM 3.8. *The discrete approximation space $\mathcal{U}_0^{(1),k}(\Omega) \subset H(\text{curl}, \Omega)$ of functions with vanishing tangential traces on $\partial\Omega$ admits a Helmholtz decomposition. That is, if $v \in \mathcal{U}_0^{(1),k}(\Omega)$, we can write $v = \nabla q + w$ with $q \in \mathcal{U}_0^{(0),k}(\Omega)$ and $w \in \mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega)$.*

Proof. If $q \in \mathcal{U}_0^{(0),k}(\Omega)$, it has zero trace on all the faces and edges of Ω . Therefore, the tangential components of ∇q are also zero on the faces and edges. We already know that grad maps $\mathcal{U}_0^{(0),k}(\Omega)$ into $\mathcal{U}^{(1),k}(\Omega)$ from Lemma 3.4, and so it is clear that grad maps $\mathcal{U}_0^{(0),k}(\Omega)$ into $\mathcal{U}_0^{(1),k}(\Omega)$. Injectivity of this map follows from Lemma 3.3. Now we count dimensions. From Section 3.1 we saw that $\dim \mathcal{U}_0^{(0),k}(\Omega) = (k-1)^3$, and from Lemma 3.7,

$$\dim \mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega) = \dim \mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega_\infty) = k(k-1)^2 + k(k-1)^2 + (k-1)^2 = (2k+1)(k-1)^2.$$

From the same lemma, we know $\text{grad} \mathcal{U}_0^{(0),k}(\Omega) \cap \mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega) = 0$. Both of these are subspaces of $\mathcal{U}_0^{(1),k}(\Omega)$. So,

$$\dim \left\{ \text{grad} \mathcal{U}_0^{(0),k}(\Omega) \cup \mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega) \right\} = (2k+1)(k-1)^2 + (k-1)^3 = 3k(k-1)^2$$

which is the dimension of $\mathcal{U}_0^{(1),k}(\Omega)$. Hence $\mathcal{U}_0^{(1),k}(\Omega) = \text{grad} \mathcal{U}_0^{(0),k}(\Omega) \oplus \mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega)$. \square

3.3. $H(\text{div}, \Omega)$ -conforming approximation spaces . In this section we shall establish that $\text{curl} \mathcal{U}^{(1),k}(\Omega) \subset \mathcal{U}^{(2),k}(\Omega)$. We then show that the div operator is injective on a certain subspace of $\mathcal{U}^{(2),k}(\Omega)$, which will be used in establishing unisolvency of the divergence-conforming elements on the pyramid. We finally demonstrate a decomposition of this discrete space.

LEMMA 3.9. *The curl operator maps elements of $\overline{\mathcal{U}^{(1),k}}(\Omega)$ into $\overline{\mathcal{U}^{(2),k}}(\Omega)$. The proof of this lemma is a calculation similar to the one in Lemma 3.4, and is omitted here.*

We now need to identify elements of $\mathcal{U}^{(2),k}(\Omega)$ which have vanishing normal traces on the faces of the finite pyramid. Denote these by $\mathcal{U}_0^{(2),k}(\Omega)$. From Lemma 1.4, we know that if $\Gamma_{i,\Omega}^2 u = 0$ for some $u \in \mathcal{U}^{(2),k}(\Omega)$, then the pull-back $\Gamma_{i,\Omega_\infty}^2 \phi^* u = 0$ on the associated face of Ω_∞ . This allows us to characterize $\mathcal{U}_0^{(2),k}(\Omega)$ easily.

LEMMA 3.10. *Functions in $\mathcal{U}_0^{(2),k}(\Omega)$ can be represented as $(\phi^{-1})^*(u)$, where $u \in \mathcal{U}_0^{(2),k}(\Omega_\infty)$ have the form*

$$\frac{z^{k-1}}{(1+z)^{k+2}} \begin{pmatrix} 2t \\ 2s \\ (1+z)(s_y + t_x) \end{pmatrix} + \begin{pmatrix} x(1-x)\chi_1 \\ y(1-y)\chi_2 \\ z\chi_3 \end{pmatrix}, \quad (3.7)$$

where $s = y(1-y)\sigma$, $t = x(1-x)\tau$, with $\chi_1 \in Q_{k+2}^{k-2,k-1,k-2}$, $\chi_2 \in Q_{k+2}^{k-1,k-2,k-2}$, $\chi_3 \in Q_{k+2}^{k-1,k-1,k-2}$, $\sigma \in Q^{k-1,k-2}(x,y)$ and $\tau \in Q^{k-2,k-1}(x,y)$

Proof.

It is easily verified that functions of the form (3.7) have vanishing normal components on the faces S_{i,Ω_∞} of the infinite pyramid; their (inverse) pullbacks to the finite pyramid will thus have vanishing normal components on the faces $S_{i,\Omega}$ of Ω . \square We note also that

$$\begin{aligned} \dim \mathcal{U}_0^{(2),k}(\Omega) &= \dim \mathcal{U}_0^{(2),k}(\Omega_\infty) \\ &= k(k-1)^2 + k(k-1)^2 + k^2(k-1) + k(k-1) + k(k-1) \\ &= 3k^3 - 3k^2. \end{aligned}$$

We now present a subspace of $\mathcal{U}_0^{(2),k}(\Omega)$ on which the divergence operator will be injective.

DEFINITION 3.11. *Define $\mathcal{U}_{0,\text{div}}^{(2),k}(\Omega) := \{v | v = (\phi^{-1})^*(u), u \in \mathcal{U}_{0,\text{div}}^{(2),k}(\Omega_\infty)\}$ where*

$$\mathcal{U}_{0,\text{div}}^{(2),k}(\Omega_\infty) := \text{span}\left\{\frac{z^{k-1}}{(1+z)^{k+2}} \begin{pmatrix} r_y + 2t \\ r_x + 2s \\ (1+z)(r_{xy} + s_y + t_x) \end{pmatrix}\right\} \oplus \text{span}\left\{\begin{pmatrix} 0 \\ 0 \\ z\chi_3 \end{pmatrix}\right\} \quad (3.8)$$

and where $r(x,y) = x(1-x)y(1-y)p(x,y)$, $p \in Q^{k-2,k-2}$, $t = x(1-x)\tilde{t}$, $\tilde{t} \in P^{k-2}(x)$, $s = y(1-y)\tilde{s}$, $\tilde{s} \in P^{k-2}(y)$, and $\chi_3 \in Q_{k+2}^{k-1,k-1,k-2}$.

LEMMA 3.12. *The divergence operator is injective on $\mathcal{U}_{0,\text{div}}^{(2),k}(\Omega)$.*

Proof. We shall show that the divergence operator is injective on $\mathcal{U}_{0,\text{div}}^{(2),k}(\Omega_\infty)$, and the desired result on Ω will follow by the properties of the pullback operator ϕ and the commutativity of ϕ with the divergence. Let u be as in (3.8). If $\nabla \cdot u = 0$, then

$$0 = \nabla \cdot u = \frac{(k-1)z^{k-2}}{(1+z)^{k+2}}(r_{xy} + t_x + s_y) + \frac{\partial}{\partial z}(z\chi_3).$$

We can factorize $\chi_3 = \sum_{i=0}^{k-2} \frac{z^i}{(1+z)^{k+2}} q_i(x,y)$. We now compare coefficients of like powers of z . Since r, t and s have no dependence on z , we obtain

$$\begin{aligned} 0 &= \frac{(k-1)z^{k-2}(r_{xy} + t_x + s_y)}{(1+z)^{k+2}} + \frac{d}{dz} \left(\sum_{i=0}^{k-2} \frac{z^{i+1} q_i(x,y)}{(1+z)^{k+2}} \right) \\ &= \frac{(k-1)z^{k-2}(r_{xy} + t_x + s_y)}{(1+z)^{k+2}} + \left(\sum_{i=0}^{k-2} \frac{z^{i+1}(i-k-1) + (1+i)z^i}{(1+z)^{k+3}} q_i(x,y) \right). \end{aligned}$$

This is impossible, unless

$$(r_{xy} + t_x + s_y) = 0, \quad q_i(x, y) = 0.$$

However, t only depends on x , and s only depends on y . From the form of r , it must be that $r \equiv 0 \equiv t \equiv s$. Therefore, $\nabla u \neq 0$ for any non-zero $u \in \mathcal{U}_{0,\text{div}}^{(2),k}(\Omega_\infty)$. Using the properties of the pullback operator, $\nabla \cdot v = 0 \Rightarrow v \equiv 0$ for all $v \in \mathcal{U}_{0,\text{div}}^{(2),k}(\Omega)$.

Finally, note that $\dim \mathcal{U}_{0,\text{div}}^{(2),k}(\Omega) = \dim \mathcal{U}_{0,\text{div}}^{(2),k}(\Omega_\infty) = (k-1)^2 + 2(k-1) + k^2(k-1) = k^3 - 1$. \square

Just as in the previous section, we can use Lemma (3.12) to exhibit a convenient decomposition of the discrete approximation space.

LEMMA 3.13. *Any $v \in \mathcal{U}_0^{(2),k}(\Omega)$ can be decomposed as $v = \nabla \times w_1 + w_2$ with $w_1 \in \mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega)$, $w_2 \in \mathcal{U}_{0,\text{div}}^{(2),k}(\Omega)$.*

Proof. Lemma 3.9 tells us that the curl operator maps $\overline{\mathcal{U}^{(1),k}}(\Omega)$ into $\overline{\mathcal{U}^{(2),k}}(\Omega)$. Observe that if the tangential components of v are zero on some surface then the component of $\nabla \times v$ that is normal to the surface will also be zero and so the curl operator maps $\mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega)$ into $\mathcal{U}_0^{(2),k}(\Omega)$. By Lemma 3.7 we know that this mapping is injective.

By construction, $\mathcal{U}_{0,\text{div}}^{(2),k}(\Omega)$ is a subset of $\mathcal{U}_0^{(2),k}(\Omega)$ and by Lemma 3.12, $\nabla \cdot w \neq 0$ for all $w \in \mathcal{U}_{0,\text{div}}^{(2),k}(\Omega)$. Hence $\mathcal{U}_{0,\text{div}}^{(2),k}(\Omega) \cap \mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega)$ is empty. We now count dimensions. We established in the proof of Lemma (3.12) that $\mathcal{U}_{0,\text{div}}^{(2),k}(\Omega)$ has dimension $k^3 - 1$ and from the previous section we know $\mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega)$ has dimension $2k^3 - 3k^2 + 1$. Thus,

$$\dim(\text{curl} \mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega) \cup \mathcal{U}_{0,\text{div}}^{(2),k}(\Omega)) = 3k^3 - 3k^2 = \dim \mathcal{U}_0^{(2),k}(\Omega),$$

which shows that $\mathcal{U}_0^{(2),k}(\Omega) = \text{curl} \mathcal{U}_{0,\text{curl}}^{(1),k}(\Omega) \oplus \mathcal{U}_{0,\text{div}}^{(2),k}(\Omega)$. This establishes the desired decomposition. \square

3.4. $L^2(\Omega)$ -conforming approximation spaces. We first note that the dimension of $\mathcal{U}^{(3),k}(\Omega) = \dim \mathcal{U}^{(3),k}(\Omega_\infty) = \dim(Q_{k+3}^{k-1,k-1,k-1}) = k^3$. It is a straightforward matter to determine that the divergence operator is well defined as a map from $\overline{\mathcal{U}^{(2),k}}(\Omega)$ to $\mathcal{U}^{(3),k}(\Omega)$. We record the result here in a lemma.

LEMMA 3.14. *The divergence operator maps elements of $\overline{\mathcal{U}^{(2),k}}(\Omega)$ into $\mathcal{U}^{(3),k}(\Omega)$.*

LEMMA 3.15. *Any element $u \in \mathcal{U}^{(3),k}(\Omega)$ can be written uniquely as*

$$u = \nabla \cdot w + \lambda, \quad w \in \mathcal{U}_{0,\text{div}}^{(2),k}(\Omega), \quad \lambda \in \mathbb{R}.$$

Proof. From Lemma 3.14, we know that $\text{div} \mathcal{U}_{0,\text{div}}^{(2),k}(\Omega) \subset \mathcal{U}^{(3),k}(\Omega)$. We also know that the constants belong to $\mathcal{U}^{(3),k}(\Omega)$. Now, $\dim(\text{div} \mathcal{U}_{0,\text{div}}^{(2),k}(\Omega)) = k^3 - 1$, which is one less than the dimension of $\mathcal{U}^{(3),k}(\Omega)$. Now, suppose we could find $w \in \text{div} \mathcal{U}_{0,\text{div}}^{(2),k}(\Omega)$ so that $\nabla w = 1$ on Ω . By definition of $\mathcal{U}_{0,\text{div}}^{(2),k}(\Omega)$, we know that w has zero normal components on the faces of Ω . From the divergence theorem, this is impossible. Hence, we have shown that the constants are not contained in $\text{div} \mathcal{U}_{0,\text{div}}^{(2),k}(\Omega)$, and therefore $\text{div} \mathcal{U}_{0,\text{div}}^{(2),k}(\Omega) \oplus \mathbb{R} = \mathcal{U}^{(3),k}(\Omega)$. This completes the proof. \square

4. First order elements on the pyramid. The approximation spaces $\mathcal{U}^{(s),k}(\Omega)$ which we constructed include the elements presented by Gradinaru and Hiptmair [12] as the special case $k = 1$. To demonstrate this, we will map the basis functions presented in that paper onto the infinite pyramid, and demonstrate that these (pulled-back) elements belong to $\mathcal{U}^{(s),k}(\Omega_\infty)$. The properties of the pullback then allow us to conclude the set inclusions on the finite pyramid. The reason for this indirect approach is the tensorial nature of the approximation spaces on Ω_∞ , which makes it easier to examine basis functions.

- **The lowest-order $H^1(\Omega)$ element:** The basis functions for the $H^1(\Omega)$ element given in equation 3.2 of [12] are denoted π_i , $i = 1..5$. Set $\tilde{\pi}_i = \phi^* \pi_i$.

$$\begin{aligned} \tilde{\pi}_1 &= \frac{(x-1)(y-1)}{1+z}, & \tilde{\pi}_2 &= \frac{x(y-1)}{1+z}, & \tilde{\pi}_3 &= \frac{(x-1)y}{1+z}, & \tilde{\pi}_4 &= \frac{xy}{1+z}, \\ \tilde{\pi}_5 &= \frac{z}{1+z}. \end{aligned} \quad \text{It is clear that } \tilde{\pi}_i \in \mathcal{U}^{(0),1}(\Omega_\infty).$$

- **The lowest-order $H(\text{curl}, \Omega)$ element:** We proceed as in the $H^1(\Omega)$ case. Set $\tilde{\gamma}_i = \phi^* \gamma_i$ where the γ_i , $i = 1..8$ are the basis functions for the curl-conforming element in [12]³:

$$\begin{aligned} \tilde{\gamma}_1 &= \frac{1}{(1+z)^2} \begin{pmatrix} 1-y \\ 0 \\ 0 \end{pmatrix}, & \tilde{\gamma}_2 &= \frac{1}{(1+z)^2} \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix}, & \tilde{\gamma}_3 &= \frac{1}{(1+z)^2} \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix}, \\ \tilde{\gamma}_4 &= \frac{1}{(1+z)^2} \begin{pmatrix} 0 \\ 1-x \\ 0 \end{pmatrix}, & \tilde{\gamma}_5 &= \frac{1}{(1+z)^2} \begin{pmatrix} z(1-y) \\ z(1-x) \\ (1-y)(1-x) \end{pmatrix}, & \tilde{\gamma}_6 &= \frac{1}{(1+z)^2} \begin{pmatrix} z(y-1) \\ zx \\ x(1-y) \end{pmatrix}, \\ \tilde{\gamma}_7 &= \frac{1}{(1+z)^2} \begin{pmatrix} zy \\ z(x-1) \\ y(1-x) \end{pmatrix}, & \tilde{\gamma}_8 &= \frac{1}{(1+z)^2} \begin{pmatrix} -zy \\ -zx \\ xy \end{pmatrix} \end{aligned} \tag{4.1}$$

These are also the pullbacks of the basis functions for the first order curl conforming element given by Graglia et al. [13]. Note that these are all edge shape functions. It is easy to see that $\tilde{\gamma}_i$ are shape functions specified in the previous section for $H_w(\text{curl}, \Omega_\infty)$ with $k = 1$.

- **The lowest-order $H(\text{div})$ element:** Set $\tilde{\zeta}_i = \phi^* \zeta_i$, where ζ_i , $i = 1..5$ are the divergence-conforming basis functions

$$\begin{aligned} \tilde{\zeta}_1 &= \frac{1}{(1+z)^3} \begin{pmatrix} 0 \\ 2(y-1) \\ z \end{pmatrix}, & \tilde{\zeta}_2 &= \frac{1}{(1+z)^3} \begin{pmatrix} 2(x-1) \\ 0 \\ z \end{pmatrix}, \\ \tilde{\zeta}_3 &= \frac{1}{(1+z)^3} \begin{pmatrix} 2x \\ 0 \\ z \end{pmatrix}, & \tilde{\zeta}_4 &= \frac{1}{(1+z)^3} \begin{pmatrix} 0 \\ 2y \\ z \end{pmatrix}, & \tilde{\zeta}_5 &= \frac{1}{(1+z)^3} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}. \end{aligned} \tag{4.2}$$

³There are minor typographical errors in [12] for two of the one-forms. Based on the preceding calculations in that paper, the correct expressions are

$$\gamma_6 = \begin{pmatrix} -z + \frac{yz}{1-z} \\ \frac{xz}{1-z} \\ x - \frac{xy}{1-z} + \frac{xyz}{(1-z)^2} \end{pmatrix}, \quad \gamma_7 = \begin{pmatrix} \frac{yz}{1-z} \\ -z + \frac{xz}{1-z} \\ y - \frac{xy}{1-z} + \frac{xyz}{(1-z)^2} \end{pmatrix}$$

For completeness, we note that $\mathcal{U}^{(3),1}(\Omega)$ consists of the constants, which map to multiples of $\frac{1}{(1+z)^4}$ on the infinite pyramid. The above collections of functions are consistent with the definitions (2.4), (2.7), (2.10) and (2.13).

5. Conclusion. In this paper we have described high-order approximation spaces on a pyramidal domain. The construction is via an infinite reference pyramid which allows for simple, tensor-product descriptions of the approximation spaces. We ensure compatibility across shared faces and edges with neighbouring tetrahedral and hexahedral elements by explicitly constraining the approximation spaces. We ended by demonstrating that the elements of Gradinaru and Hiptmair [12] can be recovered as the first order approximation spaces of our construction.

In the companion article, we shall describe shape functions and degrees of freedom to complete the definition of the high-order elements. We shall also establish exactness of the discrete spaces. Since we are working with rational basis functions, it is not immediately apparent that high-degree polynomials are contained in these spaces; we shall demonstrate this polynomial approximation property as well in the companion article.

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Appendix A. Shape functions on the infinite pyramid.

We gather here, for reference, families of shape functions associated with the vertices, edges and faces for $\mathcal{U}^{(s),k}(\Omega_\infty)$ for $s = 0, 1, 2$. We note that the shape functions presented below are not hierarchical, though a hierarchical construction is also possible.

A.1. $\mathcal{U}^{(0),k}(\Omega)$ shape functions. Since the approximation space $\mathcal{U}^{(0),k}(\Omega_\infty)$ is invariant under the rotation, $R_\infty : \Omega_\infty \rightarrow \Omega_\infty$, it is only necessary to demonstrate shape functions \tilde{F}_α for a representative vertex, vertical edge, base edge and vertical face. Then, using (1.9) and the subsequent remarks, the inverse pullback of these to the finite pyramid will also be invariant under the rotation R .

1. Vertex basis functions on Ω_∞

$$\tilde{F}_{v_1} = \left\{ \frac{(1-x)(1-y)}{(1+z)^k} \right\}, \text{ \& } \tilde{F}_{v_2}, \tilde{F}_{v_3}, \tilde{F}_{v_4} \text{ are similar and } \tilde{F}_{v_5} = \frac{z^k}{(1+z)^k}.$$

There are a total of 5 vertex shape functions in $\mathcal{U}^{(0),k}(\Omega_\infty)$.

2. Basis functions for the representative vertical edge, \tilde{e}_1 :

$$\tilde{F}_{e_1} = \left\{ \frac{(1-x)(1-y)z^a}{(1+z)^k}, \quad 1 \leq a \leq k-1 \right\}, \text{ and similar on the other vertical edges}$$

There are a total of $4(k-1)$ shape functions associated with these edges.

3. Basis functions for a representative base edge:

$$\tilde{F}_{b_1} = \left\{ \frac{(1-y)(1-x)x^a}{(1+z)^k}, \quad 1 \leq a \leq k-1 \right\}.$$

There are $4(k-1)$ such edge shape functions.

4. Basis functions for the representative vertical face S_{1,Ω_∞} :

$$\tilde{F}_{S_{1,\Omega_\infty}} = \left\{ \frac{(1-x)(1-y)x^a z^b}{(1+z)^k}, \quad a, b \geq 1, \quad a+b \leq k-1 \right\}.$$

5. Basis functions for the base face, B_{Ω_∞} :

$$\tilde{F}_B = \left\{ \frac{(1-x)(1-y)x^a y^b}{(1+z)^k}, \quad 1 \leq a, b \leq k-1 \right\}.$$

There are $(k-1)^2$ such shape functions.

A.2. $\mathcal{U}^{(1),k}(\Omega)$ shape functions. As in the previous case, we can perform this construction on the infinite pyramid and we will write $\tilde{G}_\alpha = \phi^* F_\alpha$.

1. Basis functions for the representative vertical edge, \tilde{e}_1 :

$$\tilde{G}_{e_1} = \left\{ \frac{1}{(1+z)^{k+1}} \begin{pmatrix} z(y-1) \\ z(x-1) \\ (x-1)(y-1)(1+z)^\gamma \end{pmatrix}, \quad 0 \leq \gamma \leq k-1 \right\}.$$

2. Basis functions for the representative base edge:

$$\tilde{G}_{b_1} = \left\{ \frac{1}{(1+z)^{k+1}} \begin{pmatrix} x^\gamma(1-y) \\ 0 \\ 0 \end{pmatrix}, \quad 0 \leq \gamma \leq k-1 \right\}.$$

3. Basis functions for the representative vertical (triangular) face, S_{1,Ω_∞} :

We will use three types of basis function to construct \tilde{F}_{S_1} :

$$\tilde{G}_{S_{1,1}} = \left\{ \frac{z(1-y)}{(1+z)^{k+1}} \begin{pmatrix} x^{\alpha_1}(1+z)^{\gamma_1} \\ 0 \\ 0 \end{pmatrix}, \quad \gamma_1, \alpha_1 \geq 0, \quad \gamma_1 + \alpha_1 \leq k-2 \right\},$$

$$\tilde{G}_{S_{1,2}} = \left\{ \frac{x(1-x)(1-y)}{(1+z)^{k+1}} \begin{pmatrix} 0 \\ 0 \\ x^{\alpha_2} z^{\gamma_2} \end{pmatrix}, \quad \gamma_2, \alpha_2 \geq 0, \quad \gamma_2 + \alpha_2 \leq k-3 \right\},$$

$$\tilde{G}_{S_{1,3}} = \left\{ \frac{(1-y)(1-x)x^{\alpha_3}(1+z)^{k-\alpha_3-2}}{(1+z)^{k+1}} \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix}, \quad 0 \leq \alpha_3 \leq k-2 \right\},$$

$$\tilde{G}_{S_1} = \tilde{F}_{S_{1,1}} \cup \tilde{F}_{S_{1,2}} \cup \tilde{F}_{S_{1,3}}.$$

It can be verified easily that $\tilde{G}_{S_1} \subset \mathcal{U}^{(1),k}$. The size of the space spanned by \tilde{G}_{S_1} is

$$\frac{1}{2}k(k-1) + \frac{1}{2}(k-1)(k-2) + k-1 = 2\frac{1}{2}k(k-1) = \dim(P^{k-1})^2.$$

4. Basis functions for the base face, B_{Ω_∞}

$$\begin{aligned} \tilde{G}_B = & \left\{ \frac{1}{(1+z)^{k+1}} \begin{pmatrix} y(1-y)x^{\alpha_1}y^{\beta_1} \\ 0 \\ 0 \end{pmatrix}, \quad \alpha_1 \leq k-1, \beta_1 \leq k-2 \right\} \\ & \cup \left\{ \frac{1}{(1+z)^{k+1}} \begin{pmatrix} 0 \\ x(1-x)x^{\alpha_2}y^{\beta_2} \\ 0 \end{pmatrix}, \quad \alpha_2 \leq k-2, \beta_2 \leq k-1 \right\}. \end{aligned}$$

The size of \tilde{G}_B is equal to $\dim Q^{k-1,k-2} + \dim Q^{k-2,k-1}$.

LEMMA A.1. A basis for $\mathcal{U}^{(1),k}(\Omega_\infty)$ is given by the shape functions \tilde{G}_α and functions of the form

$$u = \begin{pmatrix} y(1-y)zq_1 \\ x(1-x)zq_2 \\ x(1-x)y(1-y)q_3 \end{pmatrix} + \frac{z^{k-1}}{(1+z)^{k+1}} \begin{pmatrix} r_x z \\ r_y z \\ -r \end{pmatrix}, \quad (\text{A.1})$$

where $q \in Q_{k+1}^{k-1,k-2,k-2} \times Q_{k+1}^{k-2,k-1,k-2} \times Q_{k+1}^{k-2,k-2,k-2}$ and $r = x(1-x)y(1-y)\rho$, $\rho \in Q^{k-2,k-2}[x, y]$.

A.3. Shape functions for $\mathcal{U}^{(2),k}(\Omega)$. We will denote $\tilde{H}_\alpha = \phi^* F_\alpha$.

1. Basis functions for the representative vertical face, S_{1,Ω_∞} :

$$\tilde{H}_{S_1} = \left\{ \frac{1}{(1+z)^{k+2}} \begin{pmatrix} 0 \\ 2(1-y)x^a z^b \\ -z^k \end{pmatrix}, \quad a, b \geq 0, \quad a+b \leq k-1 \right\}.$$

The number of basis functions is $k(k-1)/2$.

2. Basis functions for the base face, B_{Ω_∞} :

$$\tilde{H}_B = \left\{ \frac{1}{(1+z)^{k+2}} \begin{pmatrix} 0 \\ 0 \\ x^a y^b \end{pmatrix}, \quad 0 \leq a, b \leq k-1 \right\}.$$

LEMMA A.2. A basis for $\mathcal{U}^{(2),k}(\Omega_\infty)$ is given by the shape functions \tilde{H}_α and functions $u \in \mathcal{U}_0^{(2),k}(\Omega_\infty)$ which have the form

$$\frac{z^{k-1}}{(1+z)^{k+2}} \begin{pmatrix} 2t \\ 2s \\ (1+z)(s_y + t_x) \end{pmatrix} + \begin{pmatrix} x(1-x)\chi_1 \\ y(1-y)\chi_2 \\ z\chi_3 \end{pmatrix}, \quad (\text{A.2})$$

$$s = y(1-y)\sigma, \quad t = x(1-x)\tau,$$

with $\chi_1 \in Q_{k+2}^{k-2,k-1,k-2}$, $\chi_2 \in Q_{k+2}^{k-1,k-2,k-2}$, $\chi_3 \in Q_{k+2}^{k-1,k-1,k-2}$ and $\sigma \in Q^{k-1,k-2}(x, y)$ and $\tau \in Q^{k-2,k-1}(x, y)$.

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